KILLARA HIGH SCHOOL

2008
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 2

General Instructions

- Reading Time- 5 minutes
- Working Time – 3 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Untidy work or work with poor logic may be penalised.

Total marks (**120**)

- Attempt Questions 1-8
- All questions are of equal value
- Start a fresh booklet of paper for each question.
- Put your name and the question number at the top of booklet.

Question 1 [Start a new booklet] 15 Marks

(a) Simplify $\dfrac{6-4i}{2i}$ [2]

(b) If $z=1-i$, find:

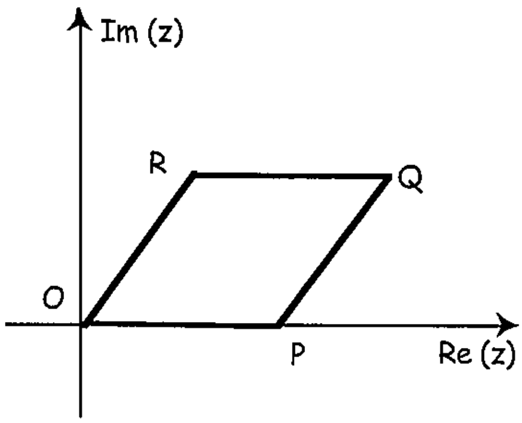
(i) $|z|$ [1]

(ii) $\arg z$ [1]

(iii) z^{-6} in the form $x+iy$ [2]

(c) Find the two square roots of i in mod-arg form. [2]

(d) OPQR is a rhombus. O lies at the origin, P on the real axis and R corresponds to the complex number $1+\sqrt{3}\,i$.



(i) Find the complex number corresponding to the point Q [2]

(ii) If the figure is rotated anticlockwise by 60° about O to form a new rhombus OP'Q'R', show this on an Argand diagram and find the complex number corresponding to the vertex at Q'. [3]

(e) Consider the following equations:

(A) $|z|-2.Im(z)>0$

(B) $z.\bar{z}-4.Re(z)>0$

(C) $Re\left(\dfrac{z-2}{z+2}\right)=0$

(D) $\arg\left(\dfrac{z-2}{z+2}\right)=\dfrac{\pi}{2}$

(E) $\arg\left(\dfrac{z-2}{z+2}\right)=-\dfrac{\pi}{2}$

[2]

Question 2 [Start a new booklet] 15 Marks

(a) Prove that $\int_0^1 \frac{(x+3)}{(x+1)(x^2+1)}dx = \frac{\pi}{2} + ln\sqrt{2}$ [3]

(b) Use the substitution $x = \frac{a}{u}$ to find $\int \frac{dx}{x\sqrt{x^2+a^2}}$ [3]

(c) (i) If $I_n = \int_0^1 x^n e^x dx$ for $n \geq 0$, prove that $I_n = e - n.I_{n-1}$ for $n \geq 1$ [2]

(ii) Find the value of I_4. [3]

(d) Given that $\frac{4x^2+3x+3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

(i) find the values of the constants A, B and C [2]

(ii) hence, or otherwise find $\int \frac{4x^2+3x+3}{(x+1)(x^2+1)}dx$ [2]

Question 3 [Start a new booklet] 15 Marks

(a) (i) Find the equation of the hyperbola on which the point P with co-ordinates $(3\sec\theta, \sqrt{7}\tan\theta)$ will always lie, and find its eccentricity. [3]

(ii) Sketch the hyperbola on a Cartesian diagram, and show on it the co-ordinates of the foci S and S', and the vertices $A(a,0)$ and $A'(-a,0)$. [2]

(iii) Show that the equation of the tangent at P on the hyperbola is $\frac{x\sec\theta}{3} - \frac{y\tan\theta}{\sqrt{7}} = 1$,

and find the co-ordinates of T, the point of intersection of the tangent and the x - axis. [5]

(iv) If P lies in such a position on the curve that TA = AS, show that the area of the triangle TSP is $\frac{\sqrt{35}}{2}$ square units. [2]

(b) If β is the acute angle between the asymptotes of the hyperbola $\frac{x^2}{r^2} - \frac{y^2}{s^2} = 1$, with eccentricity ε, $(r > s)$, find β, ε in terms of r, s and hence show that $\varepsilon = \sec\frac{\beta}{2}$. [3]

Question 4 [Start a new booklet] 15 Marks

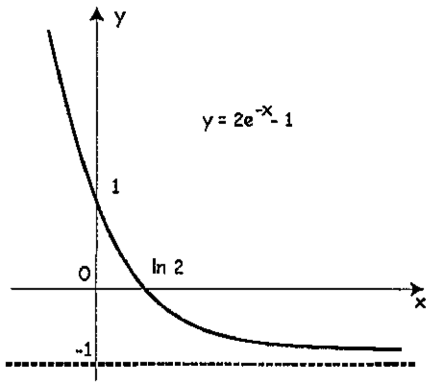
(a) On three different sets of axes, sketch the graphs in the domain $0 \le x \le 2\pi$.

(i) $y = \sin 2x$ [2]

(ii) $y = \sin^2 2x$ [2]

(iii) $y^2 = \sin 2x$ [2]

(b)



The diagram shows the graph of $f(x) = 2e^x - 1$. On separate diagrams sketch the following graphs, showing the intercepts on the axes and the equations of any asymptotes:

(i) $y = |f(x)|$. [1]

(ii) $y = \{f(x)\}^2$. [1]

(iii) $y = \dfrac{1}{f(x)}$. [2]

(iv) $y = \ln\{f(x)\}$. [1]

(c) Given that $x = \theta + \frac{1}{2}\sin 2\theta$ and $y = \theta - \frac{1}{2}\sin 2\theta$:

(i) Show that $\dfrac{dy}{dx} = \tan^2\theta$. [2]

(ii) Show that $\dfrac{d^2y}{dx^2} = \tan\theta\sec^4\theta$. [2]

Question 5 [Start a new booklet] 15 Marks

(a) When $x^3 - kx^2 - 10kx + 25$ is divided by $x - 2$ the remainder is 9.
Find the value of k. [3]

(b) If $x^3 + 4x^2 + 6x + 2 = 0$ has roots α, β, γ find the equation with roots:

(i) $\alpha^2, \beta^2, \gamma^2$ [2]

(ii) $(\alpha+1)^2, (\beta+1)^2, (\gamma+1)^2$ [2]

(c) (i) Show that if a is a multiple root of the polynomial equation $P(x) = 0$,
then $P(a) = P'(a) = 0$ [2]

(ii) The polynomial $ax^n + bx^{n-1} + 1$ is divisible by $(x-1)^2$. Find a and b in terms of n. [3]

(iii) Prove that $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots + \frac{x^n}{n!} = 0$ has no multiple roots for any $n > 1$. [3]

Question 6 [Start a new booklet] 15 Marks

(a) (i) The point P lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci S and S'.
Prove that PS + PS' = 2a [You may use a diagram as an aid]. [3]

(ii) Hence or otherwise find the Cartesian equation of the locus of all complex numbers z which satisfy the equation $|z + 3i| + |z - 3i| = 10$. [3]

(b) If 1, ω, ω^2 are the cube roots of unity:

(i) Show that $1 + \omega$ is a root of the equation $z^3 - 3z^2 + 3z - 2 = 0$ [3]

(ii) Find the integer root of this equation and the third root in terms of ω. [3]

(iii) Graph these 3 roots on an Argand diagram, and find the centre and radius of the circle on which they lie. [3]

Question 7 [Start a new booklet] 15 Marks

(a) An object is fired from the top of a cliff h metres above sea level with an initial speed V m/s and an angle of projection α. Ignore air resistance:

(i) Show that the time taken to reach the sea is given by:

$$T = \frac{1}{g}\left(V\sin\alpha + \sqrt{V^2\sin^2\alpha + 2gh}\right)$$ [4]

(ii) Hence show that if it is fired at 45° to the horizontal from the cliff, the distance from the base of the cliff at which it will land is given by:

$$D = \frac{V}{2g}\left(V + \sqrt{V^2 + 4gh}\right)$$ [2]

(iii) If the cliff is 20 m high and the object lands 50 m from the base of the cliff [with the same trajectory as in (ii)], find its initial speed. [Take $g = 10ms^{-2}$] [4]

(b) Prove by mathematical induction that:

$$2(1!) + 5(2!) + 10(3!) + 17(4!) + \ldots\ldots + (n^2+1)(n!) = n[(n+1)!]$$ [5]

Question 8 [Start a new booklet] 15 Marks

(a) The area under $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about:

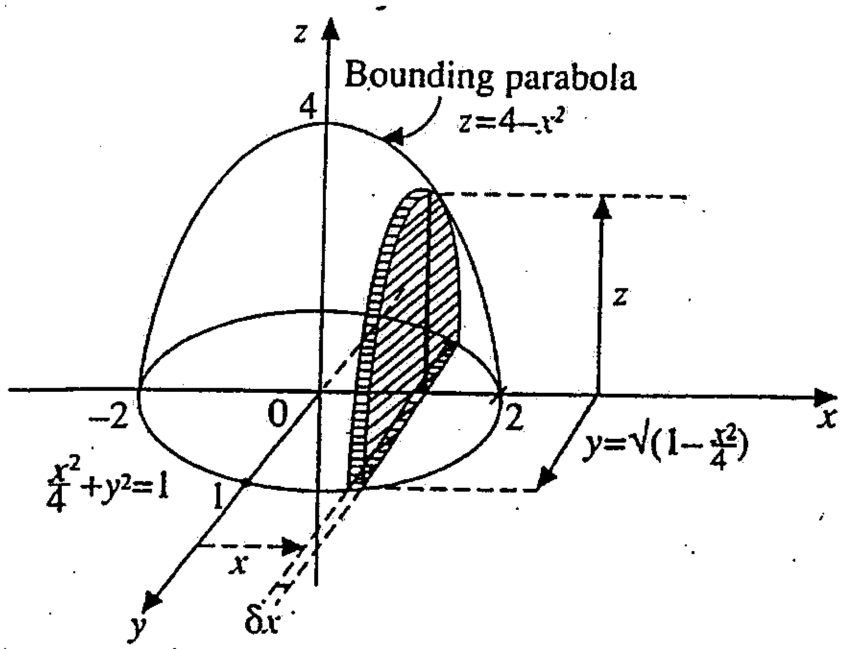
(i) the x - axis. [2]

(ii) the y - axis. [3]

(i) $x = \frac{\pi}{2}$. [4]

(b) Continued on next page........

(b) A solid has an elliptical base with semi-axes 2 and 1. Cross-sections perpendicular to the major axis of the ellipse are parabolic segments with axis passing through the major axis of the ellipse. The height of each such segment is determined by the bounding parabola of height 4, as shown in the figure below.



(i) Use Simpson's Rule to show that the area of the parabolic cross-section shown is $\frac{2}{3}\left(4-x^2\right)^{3/2}$. [3]

(ii) Write down a definite integral expression for the volume and show that its value is 4π. [3]

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

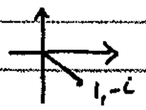
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1.

a) $\frac{6-4i}{2i} \cdot \frac{i}{i} = \frac{6i+4}{-2} = -2-3i$ ✓ [2]

b) (i) $|z| = \sqrt{2}$ [1]



(ii) $\arg z = -\frac{\pi}{4}$ [1]

(iii) $z^{-6} = \left[\sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right)\right]^{-6}$

$= \frac{1}{8}\left[\cos\left(\frac{-3\pi}{2}\right) + i\sin\left(\frac{+3\pi}{2}\right)\right]$ ✓

$= \frac{1}{8}\left[0 + i(-1)\right]$

$= -\frac{1}{8}i$ [2]

c)

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$z^2 = i \quad |z| = 1, \arg z = \frac{\pi}{4}, \frac{-3\pi}{4}$

$z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4},\ \cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)$

$= \pm\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$ [2]

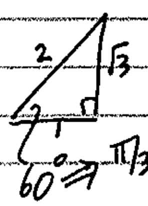
d) i) $R = 1 + \sqrt{3}i \quad |OR| = 2 \quad \therefore P = 2$

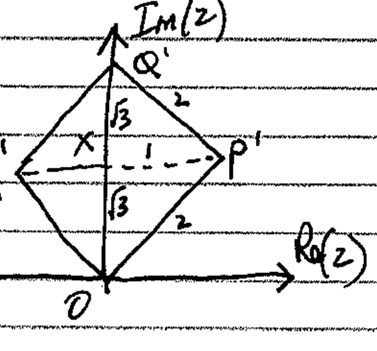
since OPQR is a Rhombus

$\overrightarrow{OQ} = 2 + 1 + \sqrt{3}i$

$= 3 + \sqrt{3}i$ [2]

(ii)





Let X lie on the midpoint of R'P'. Then it lies on the imaginary axis on the midpoint of OQ' since the diagonals bisect each other at right angles in a Rhombus.

$OQ' = \sqrt{3} + \sqrt{3} \qquad 2\sqrt{3}\,\text{cis}\,\frac{\pi}{2}$

$\therefore Q' = 2\sqrt{3}i$ [3]

e) A, D represent the same locus.

(both are semicircles with radius 1 lying above the real axis. B represents a semi-circle lying to the right of the imaginary axis; C represents a full circle, E represents a semi-circle lying below the real axis) [2]

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Question 2

a) $$\int_0^1 \frac{x+3}{(x+1)(x^2+1)}\,dx = I$$

$$\frac{x+3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$A(x^2+1)+(Bx+C)(x+1) = x+3$

$Ax^2+A+Bx^2+Cx+Bx+C$

Let $x=-1$

$2A \cancel{B} = 2$

$A = 1$

$A+B=0 \Rightarrow B=-1$

$A+C=3 \Rightarrow C=2$

so $$\frac{x+3}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{-x+2}{x^2+1}$$

$$I = \int_0^1 \left(\frac{1}{x+1} + \frac{-x+2}{x^2+1}\right)dx$$

$$= \int_0^1 \left(\frac{1}{x+1} - \frac{x}{x^2+1} + \frac{2}{x^2+1}\right)dx$$

$$= \left[\ln(x+1) + 2\tan^{-1}x - \frac{1}{2}\ln(x^2+1)\right]_0^1$$

$$= \ln 2 + 2\tan^{-1}1 - \frac{1}{2}\ln(2) - \ln 1 - 2\tan^{-1}0 - \frac{1}{2}\ln 1$$

$\Downarrow$ 0 $\quad \Downarrow$ 0 $\quad \Downarrow$ 0

$$= \ln 2 + 2\cdot\frac{\pi}{4} - \frac{1}{2}\ln 2$$

$$= \frac{1}{2}\ln 2 + \frac{\pi}{2}$$

$$= \ln\sqrt{2} + \frac{\pi}{2}.$$

[3]

b) $x = \frac{a}{u}$ $\qquad u = \frac{a}{x}$

$$dx = -\frac{a}{u^2}\,du$$

$$I = \int \frac{-a/u^2}{\frac{a}{u}\sqrt{\frac{a^2}{u^2}+a^2}}\,du$$

$$= -\int \frac{u \cdot u}{u^2}\frac{du}{\sqrt{a^2(1+u^2)}}$$

$$= -\int \frac{du}{a\sqrt{1+u^2}}$$ ✓

$$= -\frac{1}{a}\ln\left(u+\sqrt{1+u^2}\right) + C$$ ✓

$$= -\frac{1}{a}\ln\left\{\frac{a}{x} + \sqrt{1+\frac{a^2}{x^2}}\right\} + C$$

$$= -\frac{1}{a}\ln\left\{\frac{a}{x} + \frac{1}{x}\sqrt{x^2+a^2}\right\} + C$$ ✓

[3]

c)(i) $$I_n = \int_0^1 x^n e^x\,dx$$
$u \; dv$

$$= \int_0^1 x^n\,d(e^x)$$

$$= \left[x^n e^x\right]_0^1 - \int_0^1 e^x\,d(x^n)$$

$$= \left[\cancel{1}^n \cdot e^1 - 0\right] - \int_0^1 n x^{n-1} e^x\,dx$$

$$I_n = e - n\int_0^1 x^{n-1}e^x\,dx$$

$$I_n = e - n\,I_{n-1}$$

[2]

(ii) $I_4 = e - 4I_3$

$= e - 4[e - 3I_2]$

$= e - 4e + 12I_2$

Q2c(ii) cont.

$I_4 = -3e + 12(e - 2I_1)$

$= 9e - 24I_1$

$I_1 = \int_0^1 x e^x dx$

$= \int_0^1 x\, d(e^x)$

$= \left[xe^x\right]_0^1 - \int_0^1 e^x dx$

$= [e - 0] - \left[e^x\right]_0^1$

$= e - (e-1) = 1$

$I_4 = 9e - 24(1)$

$= 9e - 24$ 3

d) (i) $\frac{4x^2+3x+3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$= \frac{A(x^2+1)+(Bx+C)(x+1)}{(\ )(\ )}$

as in Q2a OR

$= \frac{Ax^2+A+Bx^2+Bx+Cx+C}{(\ )(\ )}$

$= \frac{(A+B)x^2+(B+C)x+A+C}{(\ )(\ )}$

equate coefficient

$A+B = 4$
$B+C = 3$
$A+C = 3$

$2A+2B+2C = 10$

$\underbrace{A+B}_{4}+C = 5$ $\boxed{2}$

$\therefore C = 1, A = 2, B = 2$

(ii)

$\frac{4x^2+3x+3}{(x+1)(x^2+1)} = \frac{2}{x+1} + \frac{2x+1}{x^2+1}$

$= \frac{2}{x+1} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}$

$\int \frac{4x^2+3x+3}{(x+1)(x^2+1)} = 2\ln(x+1) + \ln(x^2+1) + \frac{1}{1}\tan^{-1}x + C$

$= \ln(x+1)^2 + \ln(x^2+1) + \tan^{-1}x + C$

$= \ln\left[(x+1)^2(x^2+1)\right] + \tan^{-1}x + C$

$\boxed{2}$

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Q3

a) (i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$(a\sec\theta, b\tan\theta)$ – hyperbola.

$a = 3 \quad b = \sqrt{7}$

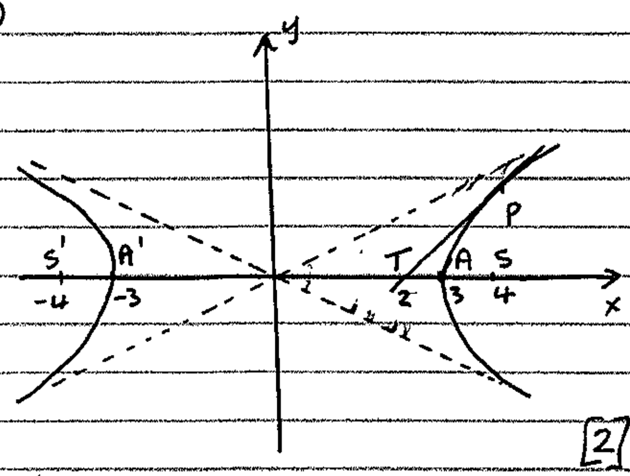
eqn is $\frac{x^2}{9} - \frac{y^2}{7} = 1$

$\left(\frac{b}{a}\right)^2 = e^2 - 1$

$\left(\frac{\sqrt{7}}{3}\right)^2 = e^2 - 1 = \frac{7}{9}$

$e^2 = \frac{16}{9} \quad e = \frac{4}{3}$ [3]

(ii)



[2]

show A $(a,0)$ $(3,0)$
A' $(-a,0)$ $(-3,0)$
S $(4,0)$
S' $(-4,0)$

(iii) $P(3\sec\theta, \sqrt{7}\tan\theta)$

$\frac{x^2}{9} - \frac{y^2}{7} = 1$

$\frac{2x}{9} - \frac{2y}{7}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2x}{9} \times \frac{7}{2y}$

$\frac{dy}{dx} = \frac{7x}{9y} = \frac{7.3\sec\theta}{9.\sqrt{7}\tan\theta}$ ✓

$= \frac{\sqrt{7}}{3} \cdot \frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{7}}{3\sin\theta}$

[could use $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$]

eqn of tangent at P

$y - \sqrt{7}\tan\theta = \frac{\sqrt{7}}{3\sin\theta}(x - 3\sec\theta)$ ✓

$3\sin\theta y - 3\sqrt{7}\sin\theta\tan\theta = \sqrt{7}x - 3\sqrt{7}\sec\theta$

now ÷ bs by $\cos\theta$

$3\tan\theta y - 3\sqrt{7}\tan^2\theta = \sqrt{7}x\sec\theta - 3\sqrt{7}\sec^2\theta$

write in general form ✓

$\sqrt{7}x\sec\theta - 3\tan\theta y = 3\sqrt{7}(\sec^2\theta - \tan^2\theta)$

$\sqrt{7}x\sec\theta - 3\tan\theta y = 3\sqrt{7}$

[OR $\frac{x\sec\theta}{3} - \frac{y\tan\theta}{\sqrt{7}} = 1$] ✓

at T $y = 0$

$\frac{x\sec\theta}{3} = 1$

$x = 3\cos\theta$ ✓

$T(3\cos\theta, 0)$ [5]

(iv) TA = AS

Focus $S(ae, 0)$ $(3.\frac{4}{3}, 0)$

$S(4,0)$

so T is $(2,0)$ ✓

Q 3 a(iv) cont.

$3\cos\theta = 2 \Rightarrow \cos\theta = 2/3$

$P(3\sec\theta \,.\, \sqrt{7}\tan\theta)$

Area of $\Delta TSP = \frac{1}{2} \times TS \times$ height

$= \frac{1}{2} \times TS \times \sqrt{7}\tan\theta$

$= \frac{1}{2} \times 2 \times \sqrt{7}\,|\tan\theta|$

but $\cos\theta = \frac{2}{3}$

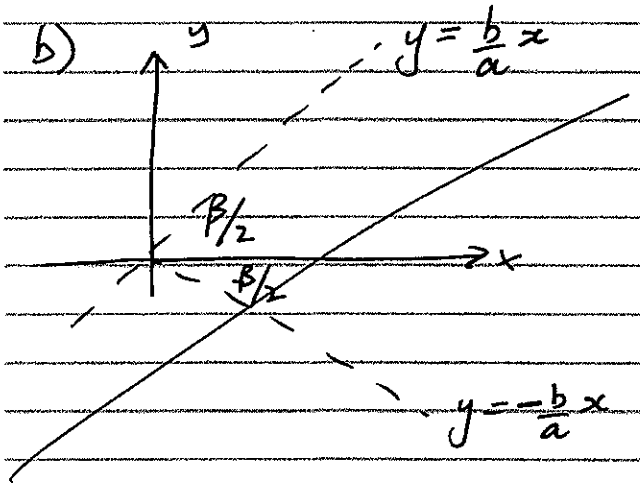
pythag.

$|\tan\theta| = \frac{\sqrt{5}}{2}$

Area of $\Delta TSP = \sqrt{7} \,.\, \frac{\sqrt{5}}{2} = \frac{\sqrt{35}}{2}$ ✓

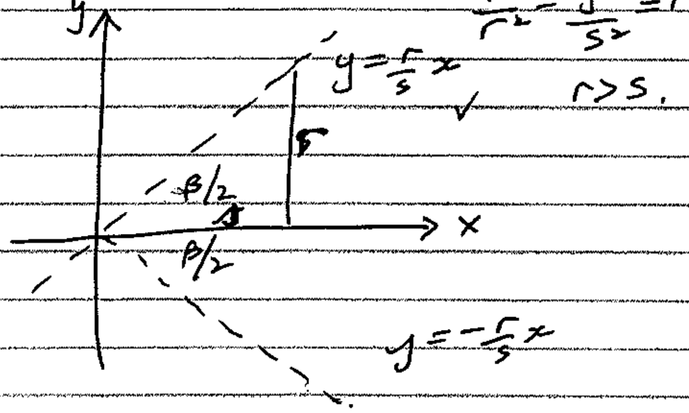
$\boxed{2}$

b)



redraw using r + s

$\frac{x^2}{r^2} - \frac{y^2}{s^2} = 1$



$\left(\frac{b}{a}\right)^2 = e^2 - 1.$

$\frac{r^2}{s^2} = e^2 - 1$

$\tan\frac{\beta}{2} = \frac{s}{r}$ ✓

$e^2 = \frac{s^2}{R^2} + 1 = \tan^2\frac{\beta}{2} + 1.$

$e^2 = \sec^2\frac{\beta}{2}$

$e > 0$ ✓

$e = \sec\frac{\beta}{2}$

$\boxed{3}$

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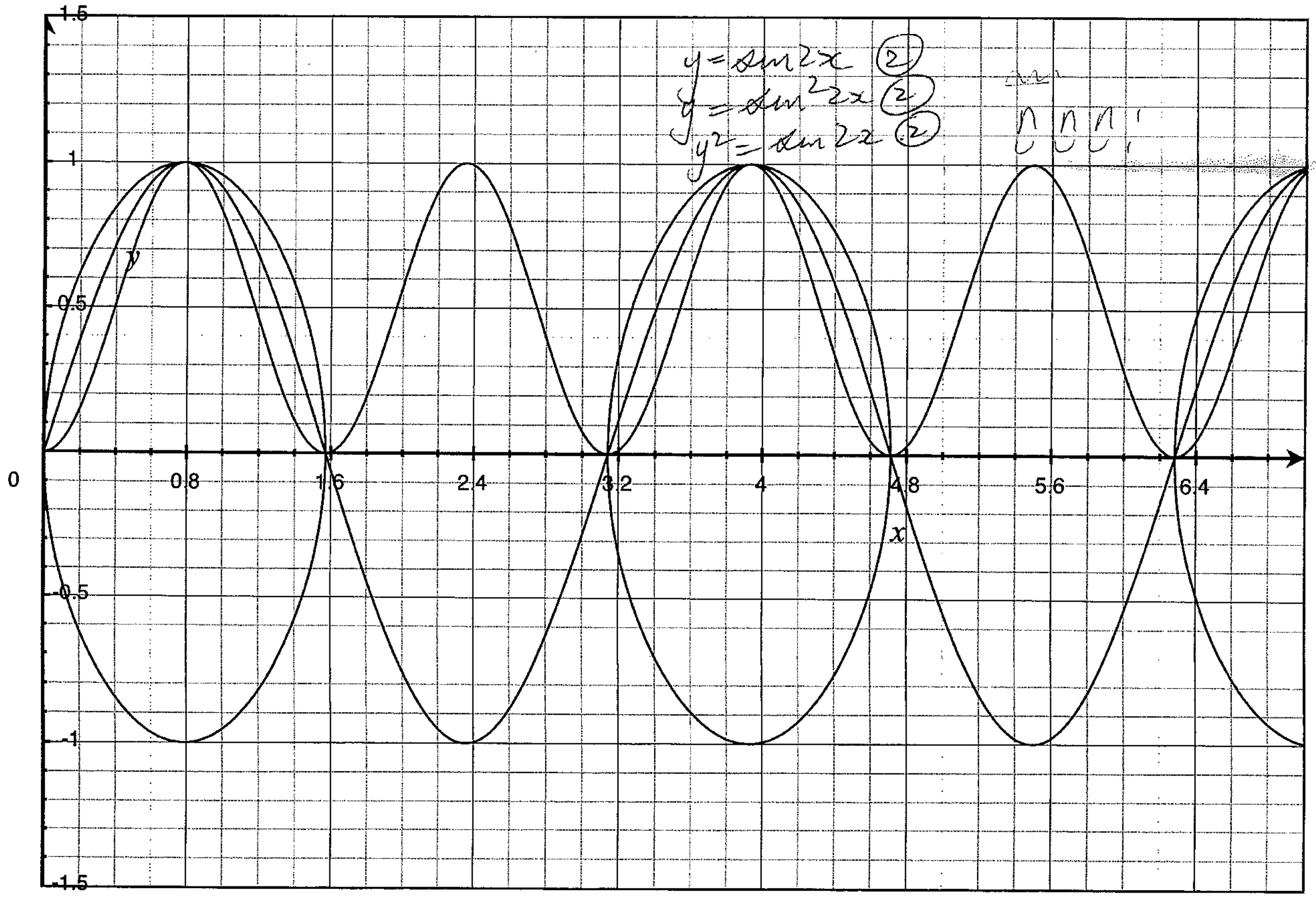
$\varepsilon = \sqrt{\frac{s^2}{r^2} + 1}$. $\beta = 2\tan^{-1}\frac{s}{R}$

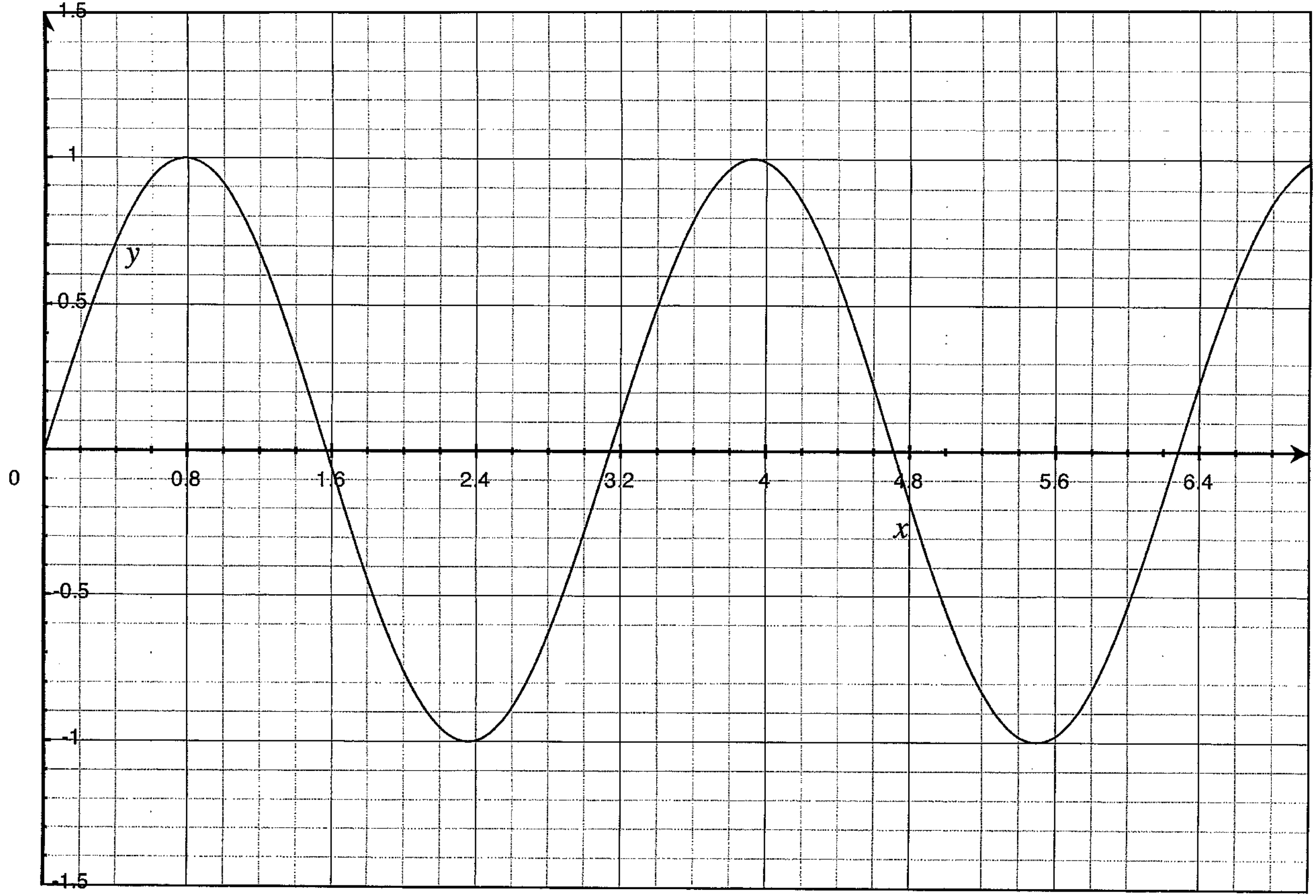
$\frac{s}{R} = \tan\frac{\beta}{2}$

$\varepsilon^2 = \tan^2\frac{\beta}{2} + 1$

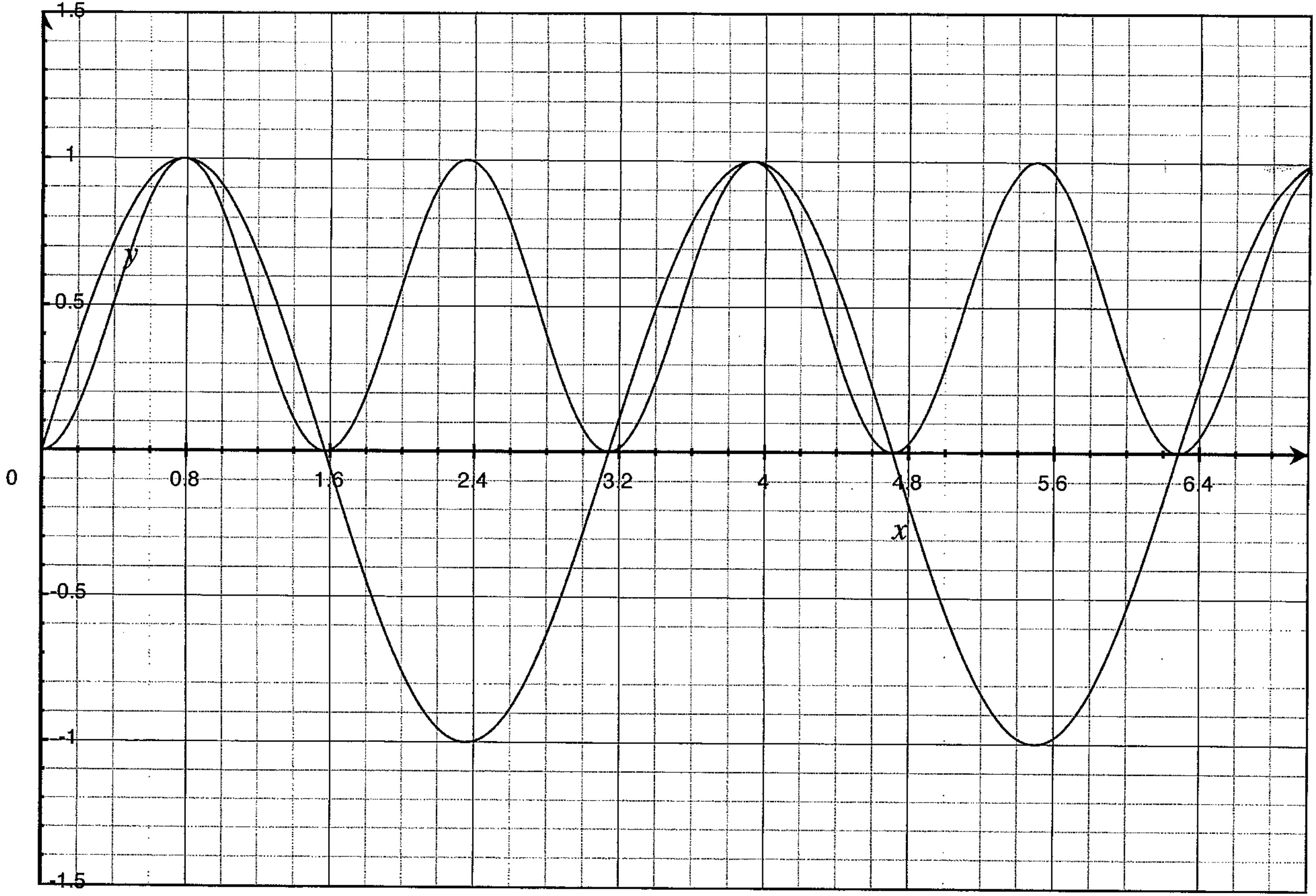
$= \sec^2\frac{\beta}{2}$

$\varepsilon = \sec\frac{\beta}{2}$.

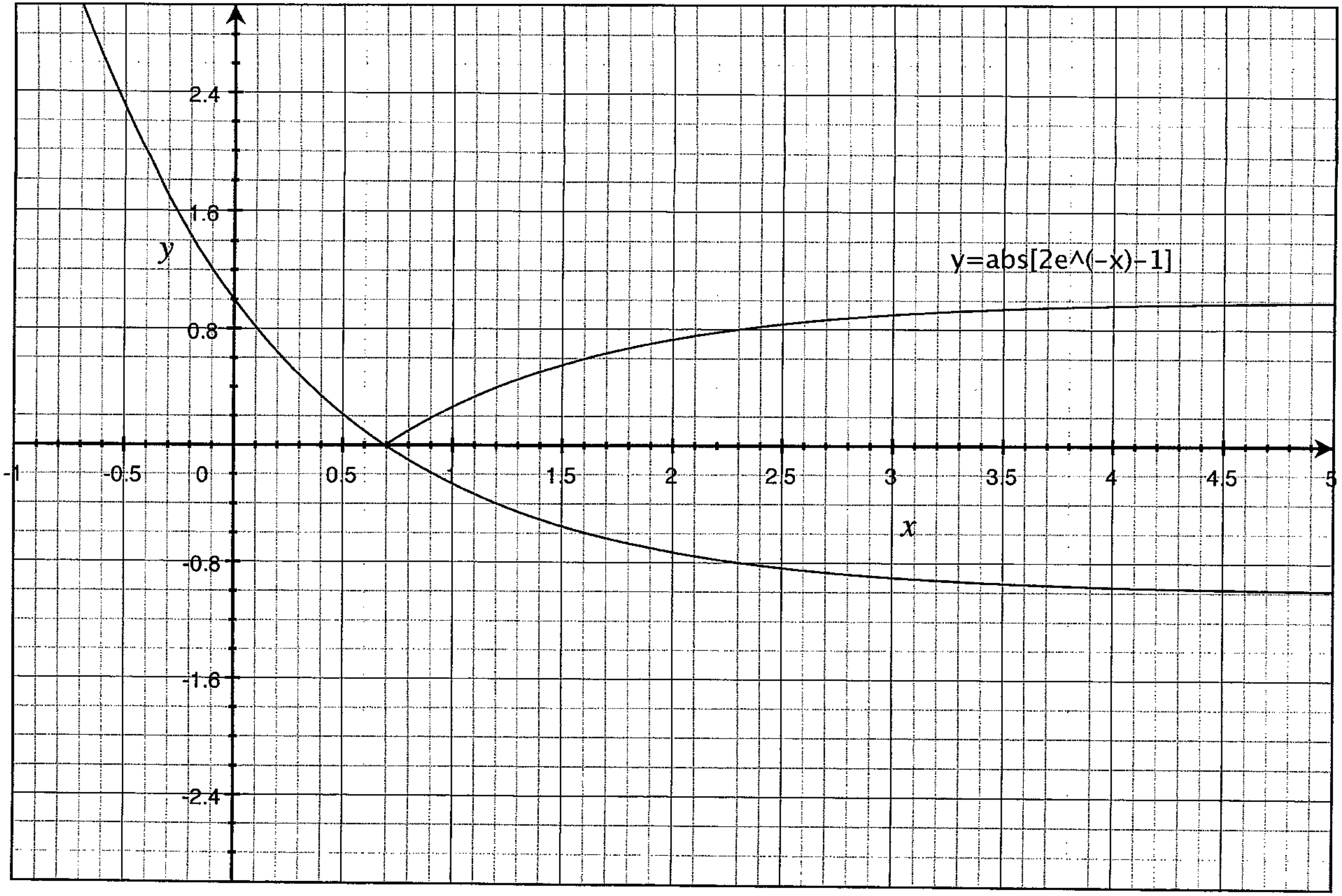

y = sin 2x (2)
y = sin² 2x (2)
y² = sin 2x (2)
1.5
1
0.5
y
-0.5
-1
-1.5
0
0.8
1.6
2.4
3.2
4
4.8
5.6
6.4
x



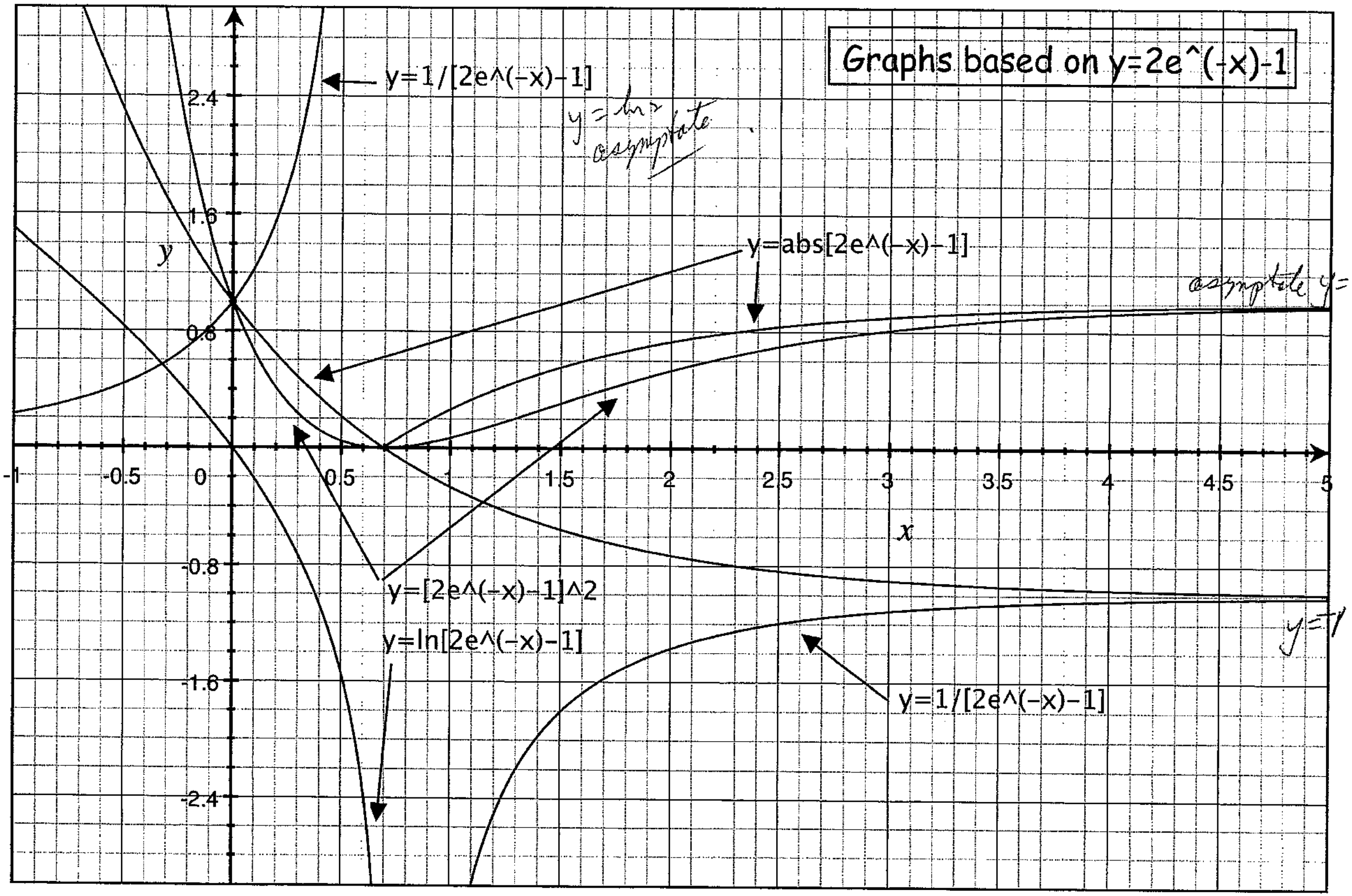
1.5
1
0.5
-0.5
-1
-1.5
0
0.8
1.6
2.4
3.2
4
4.8
5.6
6.4
y
x



1.5
1
0.5
-0.5
-1
-1.5
y
0
0.8
1.6
2.4
3.2
4
4.8
5.6
6.4
x



y=abs[2e^(-x)-1]
y
x
2.4
1.6
0.8
-0.8
-1.6
-2.4
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



Graphs based on y=2e^(-x)-1
y=1/[2e^(-x)-1]
y = ln2
asymptote
y=abs[2e^(-x)-1]
asymptote y=1
y=[2e^(-x)-1]^2
y=ln[2e^(-x)-1]
y=1/[2e^(-x)-1]
y=-1
2.4
1.6
0.8
-0.8
-1.6
-2.4
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
x
y

Q4 c. a) (6) b) (5)

$x = \theta + \frac{1}{2}\sin 2\theta$

$y = \theta - \frac{1}{2}\sin 2\theta$

$\frac{dx}{d\theta} = 1 + \frac{1}{2}\cos 2\theta \cdot 2$

$= 1 + \cos 2\theta$

$\frac{dy}{d\theta} = 1 - \cos 2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{1-\cos 2\theta}{1+\cos 2\theta}$

$= \frac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta - 1)}$

$= \frac{2\sin^2\theta}{2\cos^2\theta} = \tan^2\theta$

$\boxed{2}$

(ii) $\frac{d^2y}{dx^2} = ?$

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ $\frac{d}{d\theta}\left(\frac{dy}{dx}\right)$

$\frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx} = \left[2\tan\theta \cdot \sec^2\theta\right] \cdot \frac{1}{1+\cos 2\theta}$

$= 2\tan\theta \cdot \sec^2\theta \cdot \frac{1}{2\cos^2\theta}$

$= \frac{\tan\theta \cdot \sec^2\theta}{\cos^2\theta}$

$= \tan\theta \sec^4\theta$ (2)

/15

Q5 a) $x^3 - kx^2 - 10kx + 25 = f(x)$

$f(2) = 2^3 - 4k - 20k + 25$

$= 33 - 24k$ ✓✓

but $f(2) = 9$

$33 - 24k = 9$

$24k = 24$

$k = 1$ ✓ [3]

b)(i) $x^3 + 4x^2 + 6x + 2 = 0$

roots α, β, γ

Let

$X = \alpha^2 \qquad \alpha = \sqrt{x}$

$f(X) = (\sqrt{X})^3 + 4(\sqrt{X})^2 + 6(\sqrt{X}) + 2$

$= X\sqrt{X} + 4X + 6\sqrt{X} + 2$

$(X+6)\sqrt{X} = -2 - 4X$ ✓

square bs.

$(X+6)^2 X = 4 + 16X + 16X^2$

$(X^2 + 12X + 36)X = 4 + 16X + 16X^2$

$X^3 + 12X^2 + 36X = 4 + 16X + 16X^2$

✓

$X^3 - 4X^2 + 20X - 4 = 0$ [2]

(ii)

$X = (\alpha + 1)^2$

$\alpha + 1 = \sqrt{X} \Rightarrow \alpha = \sqrt{X} - 1$

$(\sqrt{X}-1)^3 + 4(\sqrt{X}-1)^2 + 6(\sqrt{X}-1) + 2$

$X\sqrt{X} - 3X + 3\sqrt{X} - 1 + 4X - 8\sqrt{X} + 4 + 6\sqrt{X} - 6 + 2 = 0$

$\sqrt{X}(X + 3 - 8 + 6) = 3X + 1 - 4X - 4 + 6 - 2$

$\sqrt{X}(X+1) = -X + 1$

square bs

$X(X+1)^2 = (-X+1)^2$

$X(X^2 + 2X + 1) = X^2 - 2X + 1$

$X^3 + 2X^2 + X = X^2 - 2X + 1$

$X^3 + X^2 + 3X - 1 = 0$ [2]

c)(i) Let a be a root of multiplicity n of $P(x) = 0$

Then $P(x) = (x-a)^n Q(x)$

$P(a) = (a-a)^n \cdot Q(a) = 0$

$P'(x) = Q(x) \cdot n(x-a)^{n-1} \cdot 1 + (x-a)^n \cdot Q'(x)$ ✓

$P'(a) = 0 + 0 = 0$

$P(a) = P'(a) = 0$ ✓ [2]

(ii) $P(x) = ax^n + bx^{n-1} + 1$

$P(x) = (x-1)^2 \cdot Q(x)$

since $x = 1$ is a double root

$P(1) = a + b + 1 = 0$ — (A) ✓

$P'(x) = nax^{n-1} + b(n-1)x^{n-2}$

$P'(1) = na + b(n-1) \cdot 1 = 0$ ✓

$= na + bn - b = 0$ — (B)

$n(A) \Rightarrow na + nb + n = 0$

$-b - n = 0 \quad \therefore b = -n$

Q5c(ii) sub $b=-n$ in (A)

[only 3✓]

$$a-n+1=0 \quad ✓$$
$$a=n-1 \quad ✓$$

[3]

(ii)

Assume $P(x)$ has a multiple root a for $n>1$

$$P(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots+\frac{x^n}{n!}$$

$$P'(x)=1+\frac{2x}{2!}+\frac{3x^2}{3!}+\frac{4x^3}{4!}+\dots+\frac{nx^{n-1}}{n!}$$
$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^{n-1}}{(n-1)!} \quad ✓$$

Note $P(a)=1+a+\frac{a^2}{2!}+\frac{a^3}{3!}+\frac{a^4}{4!}+\dots+\frac{a^n}{n!}=0$

This is true if $x=a$ is a root.

$$P'(a)=1+a+\frac{a^2}{2!}+\frac{a^3}{3!}+\dots\frac{a^{n-1}}{(n-1)!}$$

$$\therefore \frac{a^n}{n!}=0 \Rightarrow a^n=0 \quad ✓$$

only true if $a=0$

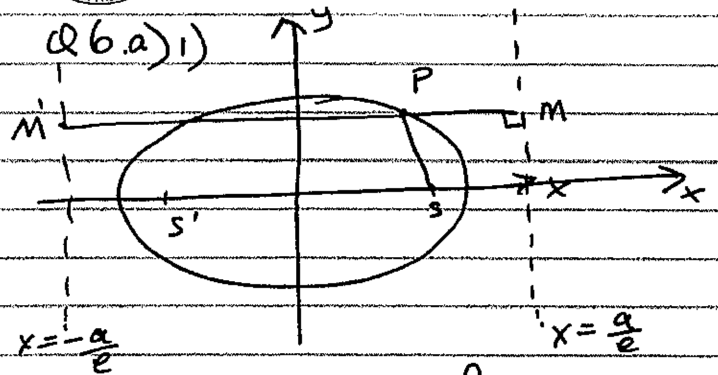
but $P(0)\neq 0$ [$=1$]

✓

$\therefore$ our assumption of multiple roots is wrong.

[3]

Q6.a) i)



P

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\qquad$ $S(ae,0)$

$S'(-ae,0)$

M, M' are on the directrices

$SP = e\,PM \qquad PS' = e\,PM'$ ✓

$PS + PS' = ePM + ePM'$

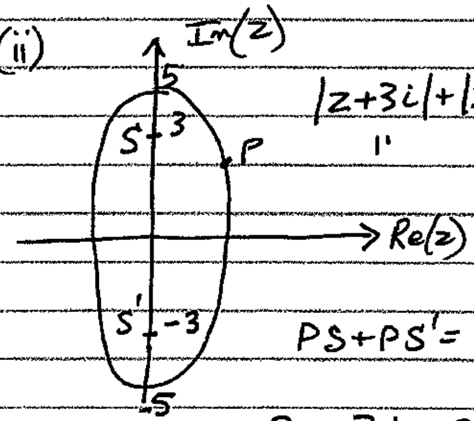
$= e(PM + PM')$ $\qquad$ +✓?

$= e(MM')$

$= e\cdot\frac{2a}{e}$ $\qquad$ 1 for a particular P

$= 2a$ ✓ $\boxed{3}$

(ii)



$|z+3i|+|z-3i|=10$

$PS + PS' = 10$

S is $3i$ $\quad$ S' is $-3i$

$a = 5$

for the ellipse $\frac{b^2}{a^2}=1-e^2$

$ae = 3$

$e = 3/5$ ✓

$\frac{b^2}{25} = 1-\frac{9}{25}$

$b^2 = 25-9 = 16$

$b = 4$ ✓

but foci are on the y axis ~~so equation is of the form~~

$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

$\frac{x^2}{16}+\frac{y^2}{25}=1$ ✓ $\boxed{3}$

1 if axes are not switched.

b) $\omega^3 = 1$

$z^3-3z^2+3z-2=0$

$z^3-3z^2+3z-1-1=0$

$(z-1)^3-1=0 \Rightarrow (z-1)^3=1$

but $\omega^3=1$

$z-1=\omega$

$z=\omega+1$ is a root

or

let $z = 1+\omega$.

$(1+\omega)^3-3(1+\omega)^2+3(1+\omega)-2 \qquad f(\omega)$

$=1+3\omega+3\omega^2+\omega^3-3-6\omega-3\omega^2$

$+3\omega+3-2$

$=1+\omega^3-2 = \omega^3-1$

but $\omega^3=1$

$\therefore$ Expression $=0$

hence $1+\omega$ is a root. $\boxed{3}$

$P(z) = z^3-3z^2+3z-2$

possible integer solutions are $\pm1, \pm2$

$P(1) = 1-3+3-2 \neq 0$

$P(-1) = -1-3-3-2 \neq 0$

$P(2) = 8-12+6-2 = 0$

$\therefore z = 2$ is a ROOT

Q6 bii

$1^3 = 1 \quad (\omega^2)^3 = 1$

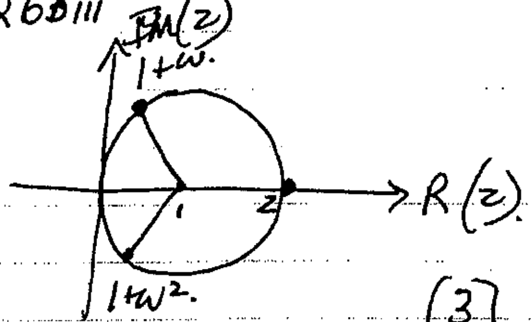
$\therefore (z-1)^3 = 1$

will have roots

$z - 1 = 1 \Rightarrow z = 2$

$z - 1 = w^2 \Rightarrow z = 1 + w^2$ [3 as before]

Q6biii



[3] each.

$1+w \quad \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$1+w^2 \quad \frac{1}{2} - \frac{\sqrt{3}}{2}i$

Q7

as required.

(b) Initially, $\ddot{x}=0$ $\ddot{y}=-g$
$\dot{x}=V\cos\alpha$ $\dot{y}=V\sin\alpha$
$x=0$ $y=h$

After t seconds, $\ddot{x}=0$ $\ddot{y}=-g$
$\dot{x}=V\cos\alpha$ $\dot{y}=V\sin\alpha-gt$
$x=Vt\cos\alpha$ $y=Vt\sin\alpha-\frac{1}{2}gt^2+h$

The time taken occurs when $y=0$,
ie $VT\sin\alpha-\frac{1}{2}gT^2+h=0$
$gT^2-2VT\sin\alpha-2h=0$

$\therefore T=\dfrac{2V\sin\alpha\pm\sqrt{4V^2\sin^2\alpha-4g(-2h)}}{2g}$

$=\frac{1}{2g}\left(2V\sin\alpha\pm\sqrt{4V^2\sin^2\alpha+8gh}\right)=\frac{1}{2g}\left(2V\sin\alpha\pm2\sqrt{V^2\sin^2\alpha+2gh}\right)$

$=\frac{1}{g}\left(V\sin\alpha+\sqrt{V^2\sin^2\alpha+2gh}\right)$ as required, since $T>0$. (4)

(ii) Now $x=Vt\cos\alpha$, $\therefore D=VT\cos\alpha=\frac{VT}{\sqrt2}$ when $\alpha=45°$

$\therefore D=\frac{V}{\sqrt2 g}\left(\frac{V}{\sqrt2}+\sqrt{\frac{V^2}{2}+2gh}\right)=\frac{V}{\sqrt2 g}\left(\frac{V+\sqrt{V^2+4gh}}{\sqrt2}\right)$

$=\frac{V}{2g}\left(V+\sqrt{V^2+4gh}\right)$ as required (2)

(iii) $h=20, D=50, g=10$

$\therefore 50=\frac{V}{20}\left(V+\sqrt{V^2+800}\right)\Rightarrow 1000=V\left(V+\sqrt{V^2+800}\right)$

$\Rightarrow \frac{1000}{V}=V+\sqrt{V^2+800}$ $\Rightarrow \frac{1000}{V}-V=\sqrt{V^2+800}$

$\Rightarrow \left(\frac{1000}{V}-V\right)^2=V^2+800$ $\Rightarrow \frac{1000000}{V^2}-2000+V^2=V^2+800$

$\Rightarrow \frac{1000000}{V^2}=2800$ $\therefore V^2=\frac{1000000}{2800}$ ie $V\doteqdot 18.9$ m/s (4)

$V=\frac{50}{\sqrt7}$

(i) Step 1
Let $n=1$
$2(1!)=2$
$1(1+1)!=2$
$\therefore$ true for $n=1$

Step 2
Assume true for $n=k$
i.e. $2(1!)+5(2!)+\dots+(k^2+1)k!$
$=k(k+1)!$
Prove true for $n=k+1$
i.e. prove that $2(1!)+5(2!)+\dots+(k^2+1)k!+[(k+1)^2+1][(k+1)!]$
$=(k+1)[(k+2)!]$

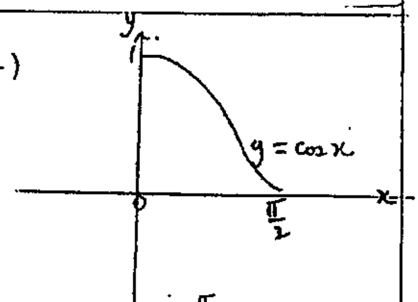
LHS $=k[(k+1)!]+[(k+1)^2+1][(k+1)!]$
$=(k+1)![k+k^2+2k+1+1]$
$=(k+1)!(k^2+3k+2)$

$=(k+1)!(k+2)(k+1)$
$=(k+2)!\ k+1$
$=(k+1)[(k+2)!]$
$=$ RHS 5

Step 3 The statement is true for $n=1$. Whenever it is true for $n=k$ it is true for $n=k+1$. Therefore the statement is true for all positive integers.

Q8

(c)



Volume $= \pi\int_0^{\frac{\pi}{2}}\cos^2 x\,dx$

$= \pi\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)dx$

$= \pi\left[\frac{1}{2}x+\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$

$= \pi\left(\frac{\pi}{4}\right)$

$= \boxed{\frac{\pi^2}{4}}\,u^3$ 2

b) By shells, the volume is $\int_0^{\frac{\pi}{2}} 2\pi xy\,dx$

$= 2\pi\int_0^{\pi/2} x\cos x\,dx$

By parts

$u = x \quad du = dx$

$dv = \cos x\,dx \quad v = \sin x$

$\int x\cos x = x\sin x - \int \sin x\,dx$

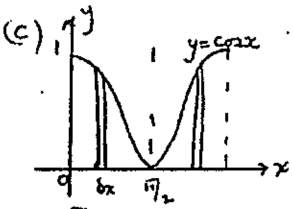
$\therefore V = \int_0^{\frac{\pi}{2}} 2\pi x\cos x\,dx$

$= 2\pi\left(x\sin x+\cos x\right]_0^{\frac{\pi}{2}}$

$= 2\pi\left(\frac{\pi}{2}-1\right)$

$= \underline{(\pi^2-2\pi)\,u^3}$ 3

(c)



Using shells, we have

$\delta V = \pi\left[\left(\frac{\pi}{2}-x+\delta x\right)^2-\left(\frac{\pi}{2}-x\right)^2\right]y$

$= \pi\left[\cancel{\left(\frac{\pi}{2}-x\right)^2}+2\left(\frac{\pi}{2}-x\right)\delta x+\cancel{(\delta x)^2}-\cancel{\left(\frac{\pi}{2}-x\right)^2}\right]y$

Since $(\delta x)^2$ can be ignored

So $V = \lim_{\delta x\to 0}\sum_{x=0}^{\frac{\pi}{2}} 2\pi\left(\frac{\pi}{2}-x\right)\cos x\,\delta x$

$= \int_0^{\frac{\pi}{2}}\pi^2\cos x - 2\pi\int_0^{\pi/2}x\cos x\,dx$

$= \pi^2\sin x\Big]_0^{\frac{\pi}{2}} - \underbrace{(\pi^2-2\pi)}_{\text{from (b)}}$

$= \pi^2-(\pi^2-2\pi)$

$= \boxed{2\pi\ \ u^3}$ 4

Volume $= \lim_{\delta x\to 0}\sum_{-2}^{2}\frac{2}{3}(4-x^2)^{3/2}dx$

$= 2\int_0^2\frac{2}{3}(4-x^2)^{3/2}$

$= \frac{4}{3}\int_0^2(4-x^2)^{3/2}dx$

Let $x = 2\sin\theta$

If $x=2$, $\theta=\frac{\pi}{2}$

If $x=0$, $\theta=0$

$dx = 2\cos\theta\,d\theta$

$\therefore \frac{4}{3}\int_0^2(4-x^2)^{3/2}dx$

$= \frac{4}{3}\int_0^{\pi/2}(4-4\sin^2\theta)^{3/2}\,2\cos\theta\,d\theta$

$= \frac{4}{3}\int_0^{\pi/2}16\cos^4\theta\,d\theta$

$= \frac{64}{3}\int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)^2d\theta$

$= \frac{64}{3}\int_0^{\pi/2}\left(\frac{1}{4}+\frac{1}{2}\cos 2\theta+\frac{1}{4}\cos^2 2\theta\right)$

$= \frac{64}{3}\int_0^{\pi/2}\left(\frac{1}{4}+\frac{1}{2}\cos 2\theta+\frac{1}{4}\left(\frac{1}{2}+\frac{1}{2}\cos 4\theta\right)\right)$

$= \frac{64}{3}\left[\frac{1}{4}\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\theta+\frac{1}{32}\sin 4\theta\right]_0$

$= \frac{64}{3}\left[\frac{\pi}{8}+\frac{\pi}{16}\right]$

$= \frac{64}{3}\cdot\frac{2\pi+\pi}{16}$

$= 4\pi\,u^3$ 3

~~16~~

15

(ii) $b = \sqrt{1-\frac{x^2}{4}} \quad a = -\sqrt{1-\frac{x^2}{4}}$

$f(a)=0 \quad f(b)=0$

$f\left(\frac{a+b}{2}\right) = 4-x^2$

$\therefore A = \frac{2\sqrt{1-\frac{x^2}{4}}}{6}\left\{0+0+4(4-x^2)\right\}$

$= \frac{1}{3}\,\frac{\sqrt{4-x^2}}{2}\;4\,(4-x^2)$

$= \frac{2}{3}(4-x^2)^{\frac{3}{2}}$ 3